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Generalized coherent states for $SU(n)$ systems

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Abstract. Generalized coherent states are developed for $SU(n)$ systems for arbitrary n. This is done by first iteratively determining explicit representations for the $SU(n)$ coherent states and then determining parametric representations useful for applications. For $SU(n)$, the set of coherent states is isomorphic to a coset space $SU(n)/SU(n-1)$, and thus shows the geometrical structure of the coset space. These results provide a convenient $(2n-1)$-dimensional space for the description of arbitrary $SU(n)$ systems. We further obtain the metric and measure on the coset space and show some properties of the $SU(n)$ coherent states.

1. Introduction

Coherent states were originally constructed and developed for the Heisenberg–Weyl group to investigate quantized electromagnetic radiation [1]. These coherent states were generated by the action of the Heisenberg–Weyl group operators on the vacuum state which led to group-theoretic generalizations by Peleromov [2] and Gilmore [3]. These two mathematical frameworks differ in some respects, such as the representations of groups and the reference states; these differences are summarized in [4]. Coherent states for $SU(2)$ have also been called atomic coherent states [5, 6], and have been found to be useful for treating atom systems, and also for investigations of quantum optical models such as nonlinear rotators [7]. $SU(2)$ coherent states have been successfully applied to the analysis of the classical limit of quantum systems, and more recently, to the investigations of nonlinear quantum systems and quantum entanglement [8, 9].

In spite of these many successful $SU(2)$ coherent state applications, not much work has been done towards generalizing the analysis to other $SU(n)$ groups, although $SU(3)$ symmetries were employed to treat a schematic nuclear shell model [10]. More recently, this lack has been addressed for $SU(3)$ systems with the explicit construction of the $SU(3)$ coherent states [11, 12], the calculation of Clebsch–Gordon coefficients [13, 14], and the investigation of Wigner functions [15]. Further, geometrical phases for $SU(3)$ systems have been discussed in [16]. These developments for $SU(3)$ are technologically useful and allow the treatment of more complex quantum systems such as coupled Bose–Einstein condensates [17].

In this paper, we construct a set of explicit coherent states for $SU(n)$, and apply group-theoretic techniques to facilitate the investigation of nonlinear quantum systems and quantum entanglement. In order to construct explicit coherent states, we need to specify the group representation and the reference states. For the chosen group representation, it is necessary to show a useful decomposition and a parametrization giving usable expressions for the coherent states. Formal approaches to the definition of coherent states are often not readily applicable. For instance, while the Baker–Campbell–Hausdorff relation derived for $SU(n)$ [18] can be used to define coherent states, this approach does not yield explicit formulae and parametrizations.

In this paper, we employ the decomposition for $SU(n)$ in [15] and exploit its symmetric parametrization. A set of coherent states of $SU(n)$ is called an orbit, and is produced by the action of group elements on a reference state which here is chosen to be the highest-weight state. For instance, for $SU(2)$ the highest-weight state for a spin-$\frac{1}{2}$ system is spin-up, and the orbit is the surface of a 3-sphere. For general n, this orbit corresponds to a $(2n-1)$-sphere, which is isomorphic to the coset space $SU(n)/SU(n-1)$. The geometrical properties of this coset space generalize the $SU(3)$ properties described in [19]. The coset space considered here, $SU(n)/SU(n-1)$, differs slightly from the coset space normally considered for coherent states, $SU(n)/U(n-1)$, by including an arbitrary phase. The coset space $SU(n)/SU(n-1)$ enables us to provide a more general method to construct coherent states. Developing the representations and decompositions of higher-rank groups rapidly becomes messy, however, the decomposition in [15] leads to a systematic procedure for the derivation of the coherent states on the coset space $SU(n)/SU(n-1)$ without additional complexity. Thence we can easily extract an arbitrary phase carrying no physical significance for application to physical systems.

In section 2, we obtain an iterative equation in $SU(n)$ coherent states for the simplest irreducible unitary representation of $SU(n)$. We also show the geometrical structure of the coset space $SU(n)/SU(n-1)$, and provide the metric and measure on the space. In section 3, our analysis is generalized to the case of finding coherent states of irreducible unitary representations for arbitrarily large dimension, and parametric representations are derived. We also show some properties of the coherent states. Finally, we summarize our results in section 4.

2. Decomposition and coset spaces for fundamental representations of $SU(n)$

In order to construct the $SU(n)$ coherent states for the fundamental $n\times n$ matrix representation, we first specify the reference state $|\phi_0\rangle$ as $(1, 0, \ldots, 0)^T$, where T denotes transpose. This state is a highest-weight state, in the sense that it is annihilated by each of the $SU(n)$ raising operators. The raising (lowering) operators J_j^h are equivalent to elementary matrices e_j^h, $h < j$ $(h > j)$ in the $n \times n$ matrix representation. Appendix A shows the commutation relations of these matrices. In this section we review the construction of $SU(2)$ coherent states, which provides the origin of the recursive relation of the $SU(n)$ coherent states. We then derive the displacement operators for $SU(3)$ and $SU(4)$, employing the $n \times n$ matrix representation of [15]. Finally, our results are extended to the $SU(n)$ case.

2.1. Review of SU(2)

Elements $g \in SU(2)$ in the fundamental 2×2 matrix representation of $SU(2)$ may be parametrized as

$$g(\theta, \varphi_1, \varphi_2) = \begin{pmatrix} \mathrm{e}^{\mathrm{i}\varphi_1}\cos\theta & -\mathrm{e}^{-\mathrm{i}\varphi_2}\sin\theta \\ \mathrm{e}^{\mathrm{i}\varphi_2}\sin\theta & \mathrm{e}^{-\mathrm{i}\varphi_1}\cos\theta \end{pmatrix} \tag{1}$$

where angles are real and lie on $0 \leqslant \theta \leqslant \pi/2$ and $0 \leqslant \varphi_1, \varphi_2 \leqslant 2\pi$, respectively. The standard approach to obtain a set of coherent states corresponding to $SU(2)/SU(1)$ begins with the decomposition of this matrix as

$$g(\theta', \varphi_1, \varphi_2) = \begin{pmatrix} \cos\frac{1}{2}\theta' & -e^{-i(\varphi_2-\varphi_1)}\sin\frac{1}{2}\theta' \\ e^{i(\varphi_2-\varphi_1)}\sin\frac{1}{2}\theta' & \cos\frac{1}{2}\theta' \end{pmatrix} \begin{pmatrix} e^{i\varphi_1} & 0 \\ 0 & e^{-i\varphi_1} \end{pmatrix} \tag{2}$$

where the new variable θ' is introduced as $\theta' = 2\theta$. The action of the right matrix on the highest-weight vector $|\phi_0\rangle = \binom{1}{0}$ changes only a phase factor and does not otherwise change the highest-weight vector. Absolute phases φ_1 and φ_2 by themselves do not carry physical significance, and only their difference is physically important. This allows the removal of an arbitrary phase, which is usually done by setting $e^{i\varphi_1} = 1$. Now the action of the left matrix on the highest-weight state gives us coherent states

$$|\boldsymbol{n}_2'\rangle = g|\phi_0\rangle = \begin{pmatrix} \cos\frac{1}{2}\theta' \\ e^{i\varphi_2}\sin\frac{1}{2}\theta' \end{pmatrix}$$

which corresponds to the 2-sphere, i.e. the surface of a three-dimensional ball, with the unit vector $(\cos\theta', e^{i\varphi_2}\sin\theta')$, or $(\cos\theta', \sin\theta'\cos\varphi_2, \sin\theta'\cos\varphi_2)$ in real coordinates, and the measure $d\mu_2' = \sin\theta'\,d\theta'\,d\varphi_2$ [2]. However, this method is not very convenient to construct coherent states for general $SU(n)$. The decomposition of $SU(n)$ equivalent to (2) is not trivial especially for larger n, and is dependent on the choice of which arbitrary phase is extracted. In this paper, to avoid using the equivalent decomposition to (2), we begin more generally with the parametrization (1), derive coherent states and then easily remove an arbitrary phase from our $SU(n)$ coherent states.

We now apply $g(\theta, \varphi_1, \varphi_2)$ of (1) to the highest-weight state $|\phi_0\rangle$. The action yields the $SU(2)$ coherent states

$$|\boldsymbol{n}_2\rangle = g|\phi_0\rangle = \begin{pmatrix} e^{i\varphi_1}\cos\theta \\ e^{i\varphi_2}\sin\theta \end{pmatrix} \tag{3}$$

which correspond to points on a 3-sphere with unit vector $(e^{i\varphi_1}\cos\theta, e^{i\varphi_2}\sin\theta)$, from which we derive the expression for the metric on the sphere

$$|ds_2|^2 = d\theta^2 + \cos^2(\theta)\,d\varphi_1^2 + \sin^2(\theta)\,d\varphi_2^2 \tag{4}$$

and the measure associated with this metric [19] as

$$d\mu_2 = \cos\theta\sin\theta\,d\theta\,d\varphi_1\,d\varphi_2. \tag{5}$$

From this set of coherent states, we now give the procedure to obtain the $SU(2)$ coherent states $|\boldsymbol{n}_2'\rangle$. This is done by setting $e^{i\varphi_1} = 1$ and introducing the variable θ' so the coherent states now correspond to a 2-sphere. This shows that one can readily remove an arbitrary phase from our $SU(n)$ coherent states without changing the decomposition of the group representation.

For the convenience of the later use, here we introduce λ-matrices and their parametrization of $SU(2)$. g may also be parametrized with using λ-matrices (i.e. Pauli matrices)

$$\lambda_1 = \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \lambda_2 = \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \qquad \lambda_3 = \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

as

$$g = e^{i\alpha\lambda_3}\,e^{i\beta\lambda_2}\,e^{i\gamma\lambda_3} \tag{6}$$

where $\varphi_1 = \alpha + \gamma$, $\varphi_2 = -\alpha + \gamma$ and $\theta = -\beta$, namely $\alpha = \frac{1}{2}(\varphi_1 - \varphi_2)$, $\beta = -\theta$ and $\gamma = \frac{1}{2}(\varphi_1 + \varphi_2)$.

2.2. Structure of $SU(n)$ for arbitrary n

Here we employ the symmetric parametrization for the $SU(n)$ matrices provided in [15] to obtain an iterative equation for the $SU(n)$ coherent states. This matrix representation efficiently yields the orbit of the highest-weight state $(1, 0, \ldots, 0)^T$, because of its symmetric decomposition. The parametrization influences the structure of the iterative equation, which we demonstrate by an example for small n. First, we derive the $SU(3)$ coherent states (which may be compared with the simplest case of [12]), and secondly the $SU(4)$ coherent states. For each example, the expression for the coherent states shows their geometrical structure and determines the metric and measure of the coset space isomorphic to the coherent states. These examples are then generalized to $SU(n)$ by determining the iterative equation for the $SU(n)$ coherent states. Lastly, we give the measure of the coset space $SU(n)/SU(n-1)$.

An arbitrary element $g \in SU(n)$ in the $n \times n$ matrix representation [15] may be parametrized as

$$g = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & & & \\ \vdots & & X_{n-1} & \\ 0 & & & \end{pmatrix} \begin{pmatrix} e^{i\varphi}\cos\theta & -\sin\theta & \\ \sin\theta & e^{-i\varphi}\cos\theta & 0 \\ & 0 & I_{n-2} \end{pmatrix} \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & & & \\ \vdots & & Y_{n-1} & \\ 0 & & & \end{pmatrix} \tag{7}$$

$$= L_{n-1} M(\theta, \varphi) R_{n-1} \tag{8}$$

where X_{n-1}, Y_{n-1} are the appropriate $(n-1) \times (n-1)$ matrices representing elements of $SU(n-1)$, and I_k is the $k \times k$ identity matrix, and we have defined three matrices L_{n-1}, M, R_{n-1} for convenience. (See appendix B for a justification of this parametrization.)

2.2.1. Structure of $SU(3)$. For $SU(3)$, since the matrices X_{n-1}, Y_{n-1} may be parametrized as (1), equation (8) gives

$$\begin{aligned} g(\varphi_1, \xi_1, \varphi_2, \varphi, \theta, \varphi_3, \xi_2, \varphi_4) &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & e^{i\varphi_1}\cos\xi_1 & -e^{-i\varphi_2}\sin\xi_1 \\ 0 & e^{i\varphi_2}\sin\xi_1 & e^{-i\varphi_1}\cos\xi_1 \end{pmatrix} \\ &\quad \times \begin{pmatrix} e^{i\varphi}\cos\theta & -\sin\theta & 0 \\ \sin\theta & e^{-i\varphi}\cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & e^{i\varphi_3}\cos\xi_2 & -e^{-i\varphi_4}\sin\xi_2 \\ 0 & e^{i\varphi_4}\sin\xi_2 & e^{-i\varphi_3}\cos\xi_2 \end{pmatrix} \\ &= L_2 M(\theta, \varphi) R_2. \end{aligned} \tag{9}$$

We take the highest-weight state $(1, 0, 0)^T$ as the reference state and now obtain the expression for the orbit. Noting that the right matrix R_2 does not change the reference state, the displacement operator for the $SU(3)$ coherent states is the product of the left and middle matrices, $L_2 M(\theta, \varphi)$. The left matrix L_2 corresponds to $SU(2)$, hence the orbit of the reference state is the coset space $SU(3)/SU(2)$. The first column of the middle matrix $M(\theta, \varphi)$ and the first and second columns of the left matrix L_2 in equation (9) can change the reference state, giving

$$|\boldsymbol{n}_3\rangle \equiv g|\phi_0\rangle = L_2 \begin{pmatrix} e^{i\varphi}\cos\theta \\ \sin\theta \\ 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} e^{i\varphi}\cos\theta \\ 0 \\ 0 \end{pmatrix}$$
$$+ \begin{pmatrix} 1 & 0 & 0 \\ 0 & e^{i\varphi_1}\cos\xi_1 & -e^{-i\varphi_2}\sin\xi_1 \\ 0 & e^{i\varphi_2}\sin\xi_1 & e^{-i\varphi_1}\cos\xi_1 \end{pmatrix} \begin{pmatrix} 0 \\ \sin\theta \\ 0 \end{pmatrix}$$
$$= \begin{pmatrix} e^{i\varphi}\cos\theta \\ 0 \\ 0 \end{pmatrix} + \sin\theta \begin{pmatrix} 0 \\ \\ |\boldsymbol{n}_2\rangle \end{pmatrix}. \tag{10}$$

The state $|\boldsymbol{n}_2\rangle$ is the $SU(2)$ coherent state given in (3), hence the coset space is isomorphic to a 5-sphere which has unit normal

$$\boldsymbol{n}_3 = (e^{i\varphi}\cos\theta, e^{i\varphi_1}\sin\theta\cos\xi_1, e^{i\varphi_2}\sin\theta\sin\xi_1) \tag{11}$$

metric

$$|ds_3|^2 = d\theta^2 + \cos^2\theta\, d\varphi^2 + \sin^2\theta(d\xi_1^2 + \cos^2\xi_1\, d\varphi_1^2 + \sin^2\xi_1\, d\varphi_2^2) \tag{12}$$

and measure

$$d\mu_3 = \cos\theta\sin^3\theta\cos\xi_1\sin\xi_1\, d\theta\, d\xi_1\, d\varphi\, d\varphi_1\, d\varphi_2. \tag{13}$$

We note here that an arbitrary phase in these coherent states for $SU(3)$ can be easily removed as discussed in subsection 2.1, and this process is also applicable to general $SU(n)$ cases.

$SU(3)$ may also be decomposed using λ-matrices [16], which yields a slightly different parametrization from (9). The $SU(2)$ λ-matrix decomposition for each matrix in (9) gives the λ-matrix expression for the $SU(3)$ coherent states. The middle matrix of (9) may be constructed using λ_2 and λ_3 as

$$M(\theta,\varphi) = e^{i\varphi/2\lambda_3}\, e^{-i\theta\lambda_2}\, e^{i\varphi/2\lambda_3}. \tag{14}$$

The left matrix of (9) may be expressed by λ_7, λ_8 and λ_3. For convenience, we define a matrix λ_8' as $(\sqrt{3}\lambda_8 - \lambda_3)/2$, which describes the $SU(2)$ diagonal generator σ_3 in the bottom right corner of the $SU(3)$ matrix $\left(\begin{smallmatrix} 1 & 0 \\ 0 & \sigma_3 \end{smallmatrix}\right)$. The left matrix is

$$\begin{aligned} L_2 &= e^{i\alpha\lambda_8'}\, e^{i\beta\lambda_7}\, e^{i\gamma\lambda_8'} \\ &= e^{i\sqrt{3}\alpha/2\lambda_8}\, e^{-i\alpha/2\lambda_3}\, e^{i\beta\lambda_7}\, e^{i\sqrt{3}\gamma/2\lambda_8}\, e^{-i\gamma/2\lambda_3} \end{aligned} \tag{15}$$

where $\varphi_1 = \alpha + \gamma$, $\varphi_2 = -\alpha + \gamma$ and $\xi_1 = -\beta$. These two expressions (14) and (15) give

$$\begin{aligned} L_2 M(\theta,\varphi) &= e^{i\alpha\lambda_8'}\, e^{i\beta\lambda_7}\, e^{i\gamma\lambda_8'}\, e^{i\varphi/2\lambda_3}\, e^{-i\theta\lambda_2}\, e^{i\varphi/2\lambda_3} \\ &= e^{i\sqrt{3}\alpha/2\lambda_8}\, e^{-i\alpha/2\lambda_3}\, e^{i\beta\lambda_7}\, e^{i\sqrt{3}\gamma/2\lambda_8}\, e^{-i\gamma/2\lambda_3}\, e^{i\varphi/2\lambda_3}\, e^{-i\theta\lambda_2}\, e^{i\varphi/2\lambda_3}. \end{aligned} \tag{16}$$

The coherent states $|\boldsymbol{n}_3\rangle$ in this representation are thus

$$\begin{aligned} |\boldsymbol{n}_3\rangle &= e^{i\alpha\lambda_8'}\, e^{i\beta\lambda_7}\, e^{i\gamma\lambda_8'}\, e^{i\varphi/2\lambda_3}\, e^{-i\theta\lambda_2}\, e^{i\varphi/2\lambda_3}|\phi_0\rangle \\ &= e^{i\sqrt{3}\alpha/2\lambda_8}\, e^{-i\alpha/2\lambda_3}\, e^{i\beta\lambda_7}\, e^{i\sqrt{3}\gamma/2\lambda_8}\, e^{-i\gamma/2\lambda_3}\, e^{i\varphi/2\lambda_3}\, e^{-i\theta\lambda_2}\, e^{i\varphi/2\lambda_3}|\phi_0\rangle. \end{aligned} \tag{17}$$

2.2.2. $SU(4)$ and $SU(n)$ for arbitrary n. Next we obtain the $SU(4)$ coherent states by applying the above procedure to the $SU(3)$ coherent states. This process shows the iterative structure of the $SU(n)$ coherent states, which allows us to define generalized coherent states for arbitrary n. An arbitrary element $g \in SU(4)$ can be factored by (8) as

$$g = L_3 M(\theta, \varphi) R_3$$

$$= \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & & & \\ \vdots & & X_3 & \\ 0 & & & \end{pmatrix} \begin{pmatrix} e^{i\varphi}\cos\theta & -\sin\theta & 0 \\ \sin\theta & e^{-i\varphi}\cos\theta & 0 \\ 0 \quad 0 & & I_2 \end{pmatrix} \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & & & \\ \vdots & & Y_3 & \\ 0 & & & \end{pmatrix} \tag{18}$$

$$= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & & \\ 0 & 0 & \multicolumn{2}{c}{X_2} \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & e^{i\varphi_1}\cos\xi_1 & -\sin\xi_1 & 0 \\ 0 & \sin\xi_1 & e^{-i\varphi_1}\cos\xi_1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\times \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & & \\ 0 & 0 & \multicolumn{2}{c}{Y_2} \end{pmatrix} M(\theta, \varphi) R_3 \tag{19}$$

where (8) has been iteratively applied twice. Here X_3, Y_3 are $SU(3)$ matrices and X_2 may be parametrized as

$$X_2 = \begin{pmatrix} e^{i\varphi_2}\cos\xi_2 & -e^{-i\varphi_3}\sin\xi_2 \\ e^{i\varphi_3}\sin\xi_2 & e^{-i\varphi_2}\cos\xi_2 \end{pmatrix}. \tag{20}$$

Taking the highest-weight state $|\phi_0\rangle = (1, 0, 0, 0)^T$ and evaluating $g|\phi_0\rangle$ as before, we observe that only two columns, the first column of X_3 and the first column of the matrix $M(\theta, \varphi)$, are important. The $SU(4)$ coherent states are

$$|\boldsymbol{n}_4\rangle = g|\phi_0\rangle = e^{i\varphi}\cos\theta \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + \sin\theta \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & & \\ 0 & 0 & \multicolumn{2}{c}{X_2} \end{pmatrix}$$

$$\times \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & e^{i\varphi_1}\cos\xi_1 & -\sin\xi_1 & 0 \\ 0 & \sin\xi_1 & e^{-i\varphi_1}\cos\xi_1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}$$

$$= \begin{pmatrix} e^{i\varphi}\cos\theta \\ 0 \\ 0 \\ 0 \end{pmatrix} + \sin\theta \begin{pmatrix} 0 \\ \\ |\boldsymbol{n}_3\rangle \\ \\ \end{pmatrix}. \tag{21}$$

We note that the matrix including Y_2 in (19) commutes with the matrix to its right, that is $[I_2 \otimes Y_2, M(\theta, \varphi)] = 0$, and does not change the state $|\phi_0\rangle$. The expression of the metric for this coset space is

$$|ds_4|^2 = d\theta^2 + \cos^2(\theta)\, d\varphi^2 + \sin^2(\theta)\{d\xi_1^2 + \cos^2(\xi_1)\, d\varphi_1^2 + \sin^2(\xi_1)(d\xi_2^2 + \cos^2(\xi_2)\, d\varphi_2^2 + \sin(\xi_2)\, d\varphi_3)\} \tag{22}$$

and the measure is

$$d\mu_4 = \cos(\theta)\sin^5(\theta)\cos(\xi_1)\sin^3(\xi_1)\cos(\xi_2)\sin(\xi_2)\,d\theta\,d\xi_1\,d\xi_2\,d\varphi\,d\varphi_1\,d\varphi_2\,d\varphi_3. \tag{23}$$

We note that the total volume is $(2\pi)^4/(6\times 4\times 2)$.

This establishes that the $SU(n)$ coherent states $|\boldsymbol{n}_n\rangle$ in this representation may be obtained from the iterative relation

$$\begin{aligned}|\boldsymbol{n}_n\rangle = g|\phi_0\rangle &= \begin{pmatrix} e^{i\varphi}\cos\theta \\ 0 \\ \vdots \\ 0\end{pmatrix} + \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & & & \\ \vdots & & X_{n-1} & \\ 0 & & & \end{pmatrix}\begin{pmatrix} 0 \\ \sin\theta \\ 0 \\ \vdots \\ 0\end{pmatrix} \\ &= \begin{pmatrix} e^{i\varphi}\cos\theta \\ 0 \\ \vdots \\ 0\end{pmatrix} + \sin\theta \begin{pmatrix} 0 \\ \\ |\boldsymbol{n}_{n-1}\rangle \\ \\ \end{pmatrix}\end{aligned} \tag{24}$$

where X_{n-1} are $SU(n-1)$ matrices, and $|\boldsymbol{n}_{n-1}\rangle$ is an $SU(n-1)$ coherent state. Since $|\boldsymbol{n}_n\rangle$ is the unit vector of the $(2n-1)$-sphere, the measure on the hypersphere is

$$\begin{aligned}d\mu_n = {}& \cos\theta\sin^{2n-3}(\theta)\cos(\xi_1)\sin^{2n-5}(\xi_1)\cdots\cos(\xi_{n-2})\sin(\xi_{n-2})\,d\theta\,d\xi_1 \\ & \cdots d\xi_{n-2}\,d\varphi\,d\varphi_1\cdots d\varphi_{n-1}.\end{aligned} \tag{25}$$

3. Arbitrary $SU(n)$ representations

We extend the results of the previous section to irreducible unitary representations of arbitrarily large dimension for $SU(n)$. We define infinitesimal operators and the basis of the group representation. Using the decomposition of $SU(n)$, we derive an iterative equation for the $SU(n)$ coherent states and further obtain its recurrence equation. For the purpose of applications, some properties of the $SU(n)$ coherent states are given.

3.1. Infinitesimal operators

We denote by T_n^N a representation of $SU(n)$ where the size number N determines the dimension of the representation. A set of simultaneous normalized eigenstates of the Cartan operators J_h^h $(1 < h \leqslant n-1)$ is employed as an appropriate basis to describe the set of coherent states. This basis will be denoted as $|m_1,\ldots,m_n\rangle$ where the m_j satisfy $N = \sum_{j=1}^n m_j$. The basis elements are also simultaneous eigenstates of the size operator $\hat{N}$ such that $\hat{N}|m_1,\ldots,m_n\rangle = N|m_1,\ldots,m_n\rangle$. For $SU(2)$, these are equivalent to the angular momentum eigenstates. The n^2-n raising operators J_j^h, $1\leqslant h<j\leqslant n$, and the same number of lowering operators J_j^h, $1\leqslant j<h\leqslant n$, of $SU(n)$ satisfy the following relations:

raising operators $(h<j)$:

$$J_j^h|m_1,\ldots,m_h,\ldots,m_j,\ldots,m_n\rangle = \sqrt{(m_h+1)m_j}\;|m_1,\ldots,m_h+1,\ldots,m_j-1,\ldots,m_n\rangle \tag{26}$$

lowering operators $(h > j)$:

$$J_j^h|m_1,\ldots,m_j,\ldots,m_h,\ldots,m_n\rangle = \sqrt{m_h(m_j+1)}\,|m_1,\ldots,m_h-1,\ldots,m_j+1,\ldots,m_n\rangle. \tag{27}$$

For Cartan operators J_h^h, $1 \leqslant h \leqslant n-1$ we have

$$J_h^h|m_1,\ldots,m_h,\ldots,m_j,\ldots,m_n\rangle = \sqrt{\frac{2}{h(h+1)}}\left(\sum_{k=1}^{h} m_k - hm_{h+1}\right) \times|m_1,\ldots,m_h,\ldots,m_j,\ldots,m_n\rangle. \tag{28}$$

3.2. $SU(n)$ coherent states

It is appropriate to choose $|\phi_0\rangle = |N,0,\ldots,0\rangle$ as the highest-weight state, since the action of any raising operator (26) on this state gives zero. The parametrization (8) shows that the representation $T_n^N(g)$ may be decomposed as

$$T_n^N(g) = T^N(L_{n-1})T^N(M)T^N(R_{n-1}). \tag{29}$$

The action of the representation $T^N(R_{n-1})$ does not change the highest-weight state as we have seen in the examples in the previous section, hence the coherent state is determined as

$$|\boldsymbol{n}_n^N\rangle = T^N(L_{n-1})T^N(M(\theta,\varphi))|N,0,\ldots,0\rangle. \tag{30}$$

The right element $T^N(M(\theta,\varphi))$ in (30) acts as an $SU(2)$ operator on the subspace $|m_1,m_2\rangle$, a cross section obtained by taking the first two elements of $|m_1,m_2,\ldots,m_n\rangle$. It is well known [2] that an arbitrary $g \in SU(2)$ may be decomposed as

$$\begin{aligned} g &= \mathrm{e}^{-\zeta^* J_2^1}\,\mathrm{e}^{-\nu J_1^1}\,\mathrm{e}^{\zeta J_1^2}\,\mathrm{e}^{\mathrm{i}\varphi_1 J_1^1} \\ &= \mathrm{e}^{\zeta J_1^2}\,\mathrm{e}^{\nu J_1^1}\,\mathrm{e}^{-\zeta^* J_2^1}\,\mathrm{e}^{\mathrm{i}\varphi_1 J_1^1}. \end{aligned} \tag{31}$$

The parameters in the above expressions correspond to the angle parameters in (1) as $\zeta = \mathrm{e}^{\mathrm{i}(\varphi_2-\varphi_1)}\tan\theta$, and $\nu = \ln\cos\theta$. Setting $\varphi_1 = \varphi$ and $\varphi_2 = 0$, $T^N(M(\theta,\varphi))$ is decomposed as (31), and acts on the subspace $|N,0\rangle$ of $|N,0,\ldots,0\rangle$ as

$$T^N(M(\theta,\varphi))|N,0,\ldots,0\rangle = \sum_{j=0}^{N} \mathrm{e}^{\mathrm{i}\varphi(N-j)}\sin^j(\theta)\cos^{N-j}(\theta)\begin{pmatrix} N \\ j \end{pmatrix}^{1/2} |N-j,j,0,\ldots,0\rangle. \tag{32}$$

The left element of the decomposition, $T^N(L_{n-1})$, does not change the first element of the state $|N-j,j,0,\ldots,0\rangle$, and acts on the subspace $|j,0,\ldots,0\rangle$ in the state $|N-j\rangle\otimes|j,0,\ldots,0\rangle$. This element acts as an $SU(n-1)$ operator on the subspace $|j,0,\ldots,0\rangle$, which generates the $SU(n-1)$ coherent states, giving

$$|\boldsymbol{n}_n^N\rangle = \sum_{j=0}^{N} \mathrm{e}^{\mathrm{i}\varphi(N-j)}\sin^j(\theta)\cos^{N-j}(\theta)\begin{pmatrix} N \\ j \end{pmatrix}^{1/2} |N-j\rangle\otimes|\boldsymbol{n}_{n-1}^j\rangle. \tag{33}$$

In order to obtain a more convenient expression for the $SU(n)$ coherent states, we derive a recurrence relation from the above iterative equation (33). The last decomposition in (31) gives the $SU(2)$ coherent states

$$\begin{aligned} |\boldsymbol{n}_2^N\rangle &= \sum_{j=0}^{N} \mathrm{e}^{\mathrm{i}j\varphi_2}\,\mathrm{e}^{\mathrm{i}(N-j)\varphi_1}\sin^j(\theta)\cos^{N-j}(\theta)\begin{pmatrix} N \\ j \end{pmatrix}^{1/2} |N-j,j\rangle \\ &= \sum_{j=0}^{N} \eta_j^N(\varphi_1,\varphi_2,\theta)|N-j,j\rangle \end{aligned} \tag{34}$$

where we define

$$\eta_j^N(\varphi_1, \varphi_2, \theta) \equiv e^{ij\varphi_2} e^{i(N-j)\varphi_1} \sin^j(\theta) \cos^{(N-j)}(\theta) \begin{pmatrix} N \\ j \end{pmatrix}^{1/2}. \tag{35}$$

The $SU(3)$ coherent states are constructed using the $SU(2)$ coherent states, and relations (33) and (35) give

$$\begin{aligned} |\boldsymbol{n}_3^N\rangle &= \sum_{j_1=0}^{N} e^{i\varphi(N-j_1)} \sin^{j_1}(\theta) \cos^{(N-j_1)}(\theta) \begin{pmatrix} N \\ j_1 \end{pmatrix}^{1/2} |N, j_1\rangle \otimes |\boldsymbol{n}_2^{j_1}\rangle \\ &= \sum_{j_1=0}^{N} \eta_{j_1}^N(\varphi, 0, \theta) \sum_{j_2=0}^{j_1} \eta_{j_2}^{j_1}(\varphi_1, \varphi_2, \xi_1) |N - j_1, j_1 - j_2, j_2\rangle \end{aligned} \tag{36}$$

in agreement with the $SU(3)$ coherent states developed in [12].

Recursively, the $SU(n)$ coherent states may be expressed by the function $\eta_k^l(\alpha, \beta, \gamma)$ of (35),

$$\begin{aligned} |\boldsymbol{n}_n^N\rangle = \sum_{j_1=0}^{N} \eta_{j_1}^N(\varphi, 0, \theta) \sum_{j_2=0}^{j_1} \eta_{j_2}^{j_1}(\varphi_1, 0, \xi_1) \cdots \sum_{j_{n-2}=0}^{j_{n-3}} \eta_{j_{n-2}}^{j_{n-3}}(\varphi_{n-3}, 0, \xi_{n-3}) \\ \times \sum_{j_{n-1}=0}^{j_{n-2}} \eta_{j_{n-1}}^{j_{n-2}}(\varphi_{n-2}, \varphi_{n-1}, \xi_{n-2}) \, |N - j_1, j_1 - j_2, \ldots, j_{n-1}\rangle. \end{aligned} \tag{37}$$

3.3. Properties of the SU(n) coherent states

For the purpose of applications, here we describe some fundamental properties of the $SU(n)$ coherent states.

(a) *Stereographic coordinates.* The decomposition (31) implies that the $SU(n)$ coherent states may be represented in the complex numbers ζ_k such that $\zeta_k = e^{i(\varphi_{k+1}-\varphi_k)} \tan(\xi_k)$. A routine change of variables gives the $SU(n)$ coherent states in this stereographic coordinates as

$$\begin{aligned} |\boldsymbol{n}_n^N\rangle = e^{i\varphi N} \left(\frac{1}{1+|\zeta|^2}\right)^N \sum_{j_1=0}^{N} (\zeta)^{j_1} \begin{pmatrix} N \\ j_1 \end{pmatrix}^{1/2} \left(\frac{1}{1+|\zeta_1|^2}\right)^{j_1} \\ \times \sum_{j_2=0}^{j_1} (\zeta_1)^{j_2} \begin{pmatrix} j_1 \\ j_2 \end{pmatrix}^{1/2} \left(\frac{1}{1+|\zeta_2|^2}\right)^{j_2} \\ \vdots \\ \times \sum_{j_{n-1}=0}^{j_{n-2}} (\zeta_{n-2})^{j_{n-1}} \begin{pmatrix} j_{n-2} \\ j_{n-1} \end{pmatrix}^{1/2} |N - j_1, \ldots, j_{n-2} - j_{n-1}, j_{n-1}\rangle. \end{aligned} \tag{38}$$

(b) *Resolution of unity.* The set of coherent states provides a resolution of unity in the coset space as

$$\frac{(N+n-1)!}{2\pi^n N!} \int d\mu_n \, |\boldsymbol{n}_n^N\rangle\langle\boldsymbol{n}_n^N| = \hat{I}. \tag{39}$$

The matrix $|\boldsymbol{n}_n^N\rangle\langle\boldsymbol{n}_n^N|$ may be expanded in terms of matrices $|N - j_1', \ldots, j_{n-1}'\rangle\langle N - j_1, \ldots, j_{n-1}|$ by using the expansion (37). The integrals with respect to ξ_k are carried out by change of integral variables $x \equiv \cos^2(\xi_k)$ allowing

$$\int_0^{\pi/2} d\xi_k \cos^{(2(m-n)+1)}(\xi_k) \sin^{(2n+1)}(\xi_k) = \frac{1}{2} \frac{n!(m-n)!}{(m+1)!}$$

while the integrals in terms of φ_j produce delta functions. The result of the all integrals cancels with the normalization factor, and gives us

$$\sum_{j_1=0}^{N}\cdots\sum_{j_{n-1}=0}^{j_{n-2}}|N-j_1,\ldots,j_{n-1}\rangle\langle N-j_1,\ldots,j_{n-1}|=\hat{I}.$$

(c) *Overlap of two coherent states.* The overlap of two coherent states may be calculated from (33), as

$$\langle \boldsymbol{n}'^{N}_{n}|\boldsymbol{n}^{N}_{n}\rangle=\left(\mathrm{e}^{\mathrm{i}(\varphi_{n-1}-\varphi'_{n-1})}\prod_{k=0}^{n-2}\sin\xi_k\sin\xi'_k\right.$$
$$\left.+\sum_{m=0}^{n-2}\left[\mathrm{e}^{\mathrm{i}(\varphi_m-\varphi'_m)}\cos\xi_m\cos\xi'_m\prod_{k=0}^{m-1}\sin\xi_k\sin\xi'_k\right]\right)^{N} \tag{40}$$

where we have changed the notation of angles, replacing θ with ξ_0, and φ with φ_0, and where we have defined $\prod_{k=0}^{-1}\sin\xi_k\sin\xi'_k=1$.

4. Summary

In conclusion, we have described $SU(n)$ coherent states for irreducible unitary representations for arbitrarily large dimension and some examples for small n have been demonstrated. The geometric structure of the $SU(n)$ coherent states has been represented using spherical coordinates. We also gave expressions for the resolution of unity, and the non-orthogonality of the coherent states. It was shown the $SU(n)$ coherent states may be derived recursively from $SU(2)$ coherent states.

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Appendix A. λ-matrices

In general, $SU(n)$ generators can be represented by n^2-n off-diagonal matrices and $n-1$ diagonal matrices. For example, $SU(4)$ has 15 generators which can be constructed using 12 off-diagonal matrices and three diagonal matrices [20]. We take $\{e^h_j\}$ as a basis for the group $SU(n)$, where e^h_j are elementary matrices. We also define e^h_j $(h<j)$ as raising operators, and e^h_j $(h>j)$ as lowering operators, respectively. Non-diagonal elements of this basis are

$$\{\beta^h_j=-\mathrm{i}(e^h_j-e^j_h),\ \Theta^h_j=e^h_j+e^j_h,\ 1\leqslant h<j\leqslant n\}. \tag{A1}$$

The commutation relations of these non-diagonal elements are

$$[\beta^h_j,\Theta^k_e]=-\mathrm{i}\delta^k_j\Theta^h_e+\mathrm{i}\delta^h_e\Theta^k_j+\mathrm{i}\delta^k_h\Theta^j_e-\mathrm{i}\delta^j_e\Theta^k_h. \tag{A2}$$

The diagonal elements $\{\eta^m_m|1\leqslant m\leqslant n-1\}$ are

$$\eta^m_m=\sqrt{\frac{2}{m(m+1)}}\left(\sum_{j=1}^{m}e^j_j-m\ e^{m+1}_{m+1}\right). \tag{A3}$$

For instance, in $SU(4)$ the 15 λ-matrices are numbered as

$$\begin{array}{lllll} \lambda_1 = \Theta_2^1 & \lambda_2 = \beta_2^1 & \lambda_3 = \eta_1^1 & & \\ \lambda_4 = \Theta_3^1 & \lambda_5 = \beta_3^1 & \lambda_6 = \Theta_3^2 & \lambda_7 = \beta_3^2 & \lambda_8 = \eta_2^2 \\ \lambda_9 = \Theta_4^1 & \lambda_{10} = \beta_4^1 & \lambda_{11} = \Theta_4^2 & \lambda_{12} = \beta_4^2 & \\ \lambda_{13} = \Theta_4^3 & \lambda_{14} = \beta_4^3 & \lambda_{15} = \eta_3^3. & & \end{array} \tag{A4}$$

These λ-matrices are the generators of the representation T_4^1.

These $SU(4)$ generators allow another expression for the coherent states. Defining a matrix λ'_{15} as $(\sqrt{6}\lambda_{15} - \sqrt{3}\lambda_8)/3$, the decomposition of $SU(4)$ using λ-matrices gives expressions for the coherent states

$$\begin{aligned} |\boldsymbol{n}_4\rangle &= \mathrm{e}^{\mathrm{i}\alpha\lambda'_{15}}\,\mathrm{e}^{\mathrm{i}\beta\lambda_{14}}\,\mathrm{e}^{\mathrm{i}\gamma\lambda'_{15}}\,\mathrm{e}^{\mathrm{i}\varphi_1/2\lambda'_8}\,\mathrm{e}^{-\mathrm{i}\xi_1\lambda_7}\,\mathrm{e}^{\mathrm{i}\varphi_1/2\lambda'_8}\,\mathrm{e}^{\mathrm{i}\varphi/2\lambda_3}\,\mathrm{e}^{-\mathrm{i}\theta\lambda_2}\,\mathrm{e}^{\mathrm{i}\varphi/2\lambda_3}|\phi_0\rangle \\ &= \mathrm{e}^{\mathrm{i}\sqrt{6}\alpha/3\lambda_{15}}\,\mathrm{e}^{-\mathrm{i}\sqrt{3}\alpha/3\lambda_8}\,\mathrm{e}^{\mathrm{i}\beta\lambda_{14}}\,\mathrm{e}^{\mathrm{i}\sqrt{6}\gamma/3\lambda_{15}}\,\mathrm{e}^{-\mathrm{i}\sqrt{3}\gamma/3\lambda_8}\,\mathrm{e}^{\mathrm{i}\sqrt{3}\varphi_1/4\lambda_8}\,\mathrm{e}^{-\mathrm{i}\varphi_1/4\lambda_3}\,\mathrm{e}^{-\mathrm{i}\xi_1\lambda_7} \\ &\qquad \times\mathrm{e}^{\mathrm{i}\sqrt{3}\varphi_1/4\lambda_8}\,\mathrm{e}^{-\mathrm{i}\varphi_1/4\lambda_3}\,\mathrm{e}^{\mathrm{i}\varphi/2\lambda_3}\,\mathrm{e}^{-\mathrm{i}\theta\lambda_2}\,\mathrm{e}^{\mathrm{i}\varphi/2\lambda_3}|\phi_0\rangle \end{aligned} \tag{A5}$$

where $\varphi_2 = \alpha + \gamma$, $\varphi_3 = -\alpha + \gamma$ and $\xi_2 = -\beta$. These expressions have been obtained directly from (16) and (19), and show the displacement operator for the $SU(4)$ coherent states.

Appendix B. The symmetric parametrization for $SU(n)$

Here we show a brief proof of the parametrization (8). A proof of this parametrization for $n = 3$ was given in [15]. Showing any element of $g \in SU(n)$ can be transformed into $R_{n-1}^{\dagger}$, we give the parametrization (8) as an inverse equation in terms of the element g. We first review the proof for $n = 3$, and prove (8) for arbitrary n inductively.

(a) For $n = 3$, an arbitrary element

$$g = \begin{pmatrix} x_{11} & x_{12} & x_{13} \\ x_{21} & x_{22} & x_{23} \\ x_{31} & x_{32} & x_{33} \end{pmatrix}$$

can be transformed as

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & a^* & b^* \\ 0 & -b & a \end{pmatrix}\begin{pmatrix} x_{11} & x_{12} & x_{13} \\ x_{21} & x_{22} & x_{23} \\ x_{31} & x_{32} & x_{33} \end{pmatrix} = \begin{pmatrix} x_{11} & x_{12} & x_{13} \\ r_2^3 & \dfrac{\sum_{k=2}^3 x_{k1}^* x_{k2}}{r_2^3} & \dfrac{\sum_{k=2}^3 x_{k1}^* x_{k3}}{r_2^3} \\ 0 & \star & \star \end{pmatrix} \tag{B1}$$

where $a = x_{21}/r_2^3$, $b = x_{31}/r_2^3$ and $r_p^q = \sqrt{\sum_{k=p}^q |x_{k1}|^2}$. Applying a matrix

$$\begin{pmatrix} x_{11}^* & \sqrt{1-|x_{11}|^2} & 0 \\ -\sqrt{1-|x_{11}|^2} & x_{11} & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

from the left on the above matrix (B1) gives

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & \star & \star \\ 0 & \star & \star \end{pmatrix}$$

where we used the constraints on g, which are $\sum_{j=1}^{3} x_{jk}^* x_{jl} = \delta_{kl}$. With suitably chosen parameters, the inversion of the above relation gives the devised SU(3) parametrization (9). Now we extend this procedure to the general result, and prove it inductively.

(b) We assume the result in the case (a), that is, for $n = m$ an arbitrary element $g \in SU(m)$ can be parametrized as (8), and for any g a matrix $X_{m-1} \in SU(m-1)$ exists such that

$$\begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & & & \\ \vdots & & X_{m-1}^\dagger & \\ 0 & & & \end{pmatrix} \begin{pmatrix} x_{11} & x_{12} & \cdots & x_{1m} \\ x_{21} & x_{22} & \cdots & x_{2m} \\ \vdots & \vdots & \vdots & \vdots \\ x_{m1} & x_{m2} & \cdots & x_{mm} \end{pmatrix} = \begin{pmatrix} x_{11} & x_{12} & \cdots & x_{1m} \\ r_2^m & \frac{\sum_{k=2}^{m} x_{k1}^* x_{k2}}{r_2^m} & \cdots & \frac{\sum_{k=2}^{m} x_{k1}^* x_{km}}{r_2^m} \\ 0 & \star & \star & \star \\ \vdots & \vdots & \vdots & \vdots \\ 0 & \star & \star & \star \end{pmatrix}. \tag{B2}$$

(c) For $n = m + 1$, using (B2), an arbitrary matrix $g \in SU(m + 1)$ can be transformed as

$$\begin{pmatrix} I_2 & 0 & \cdots & 0 \\ 0 & & & \\ \vdots & & Y_{m-1}^\dagger & \\ 0 & & & \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & \\ 0 & \frac{x_{21}^*}{r_2^{m+1}} & \frac{r_3^{m+1}}{r_2^{m+1}} & 0 \\ 0 & -\frac{r_3^{m+1}}{r_2^{m+1}} & \frac{x_{21}}{r_2^{m+1}} & \\ & 0 & & I_{m-2} \end{pmatrix} \begin{pmatrix} I_2 & 0 & \cdots & 0 \\ 0 & & & \\ \vdots & & X_{m-1}^\dagger & \\ 0 & & & \end{pmatrix}$$
$$\times \begin{pmatrix} x_{11} & x_{12} & \cdots & x_{1m+1} \\ x_{21} & x_{22} & \cdots & x_{2m+1} \\ \vdots & \vdots & \vdots & \vdots \\ x_{m+11} & x_{m+12} & \cdots & x_{m+1m+1} \end{pmatrix} = \begin{pmatrix} x_{11} & x_{12} & \cdots & x_{m+1} \\ r_2^{m+1} & \frac{\sum_{k=2}^{m+1} x_{k1}^* x_{k2}}{r_2^{m+1}} & \cdots & \frac{\sum_{k=2}^{m+1} x_{k1}^* x_{km+1}}{r_2^{m+1}} \\ 0 & \star & \cdots & \star \\ \vdots & \vdots & \vdots & \vdots \\ 0 & \star & \cdots & \star \end{pmatrix} \tag{B3}$$

where I_k are $k \times k$ identity matrices. Using the constraints for $SU(m + 1)$ matrices, $\sum_{j=1}^{m+1} x_{jk}^* x_{jl} = \delta_{kl}$, the matrix on the right-hand side can be transformed to contain an

$SU(m)$ matrix as

$$\begin{pmatrix} x_{11}^* & \sqrt{1-|x_{11}|^2} & 0 \\ -\sqrt{1-|x_{11}|^2} & x_{11} & 0 \\ 0 & & I_{m-1} \end{pmatrix}$$
$$\times \begin{pmatrix} x_{11} & x_{12} & \cdots & x_{m+1} \\ r_2^{m+1} & \frac{\sum_{k=2}^{m+1} x_{k1}^* x_{k2}}{r_2^{m+1}} & \cdots & \frac{\sum_{k=2}^{m+1} x_{k1}^* x_{km+1}}{r_2^{m+1}} \\ 0 & \star & \cdots & \star \\ \vdots & \vdots & \vdots & \vdots \\ 0 & \star & \cdots & \star \end{pmatrix}$$
$$= \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & & & \\ \vdots & & Y_m & \\ 0 & & & \end{pmatrix}. \tag{B4}$$

Since the second matrix can be parametrized equivalently to the $n = m$ case, the product of the three matrices constructs an $SU(n)$ matrix. The inversion of this relation gives

$$\begin{pmatrix} x_{11} & x_{12} & \cdots & x_{1\,m+1} \\ x_{21} & x_{22} & \cdots & x_{2\,m+1} \\ \vdots & \vdots & \vdots & \vdots \\ x_{m+1\,1} & x_{m+1\,2} & \cdots & x_{m+1\,m+1} \end{pmatrix} = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & & & \\ \vdots & & X_m & \\ 0 & & & \end{pmatrix}$$
$$\times \begin{pmatrix} x_{11} & -\sqrt{1-|x_{11}|^2} & \\ \sqrt{1-|x_{11}|^2} & x_{11}^* & 0 \\ & 0 & I_{m-1} \end{pmatrix} \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & & & \\ \vdots & & Y_m & \\ 0 & & & \end{pmatrix}. \tag{B5}$$

The elements, x_{11} and $\sqrt{1-|x_{11}|^2}$, can be parametrized as

$$x_{11} = \mathrm{e}^{\mathrm{i}\varphi}\cos\theta \qquad \sqrt{1-|x_{11}|^2} = \sin\theta. \tag{B6}$$

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